Game Theory

6 – Dynamic Games with imperfect information
Review of lecture five

- Game tree and strategies
- Dynamic games of perfect information
- Games and subgames
- Subgame perfect Nash equilibrium
- Dynamic games in normal form
- Backward induction
Perfect information

• Dynamic games are games where players move in sequence
• If all players know at each stage of the game the entire development of the game before the current move, dynamic games are games of perfect information
Imperfect information in dynamic games

Three analytically equivalent definitions involving the rules of the game:

1. At some stage of the game some players do not know its entire development before the current move.
2. Players may move simultaneously.
3. Some players at some stages may move while not being observed by the others.
Static games and imperfect information

• All games in normal form considered so far (static games) are games of imperfect information

• When put them in extensive form it is necessary to indicate the simultaneity of players’ moves on the game tree

• That means that some players do not know from which node of the game tree they are making the move
Matching pennies in extensive form

Two players: A and B own a coin each, turned secretly on head or tail
Confronting coins, if both show the same face
A takes both; otherwise B takes both

<table>
<thead>
<tr>
<th></th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>head</td>
</tr>
<tr>
<td>A</td>
<td>head</td>
</tr>
<tr>
<td></td>
<td>tail</td>
</tr>
</tbody>
</table>
If we try to put this game in extensive form

If we try to put this game in extensive form, the game becomes a stupid one, where Bert can look at the preceding Ann’s move, turn his coin at the opposite face, and win systematically.

We need to indicate on the game tree that A and B move simultaneously.
A graphical convention

In games represented in extensive form (game tree) the nodes are connected with a dashed line if the moving player does not know from which node of the game tree he is making the move.

That means that the preceding player has moved secretly.

This game tree is equivalent to the table of the normal form:

- \((1, -1)\) A wins
- \((-1, 1)\) B wins
- \((-1, 1)\) B wins
- \((1, -1)\) A wins
Information set

• The set of nodes that a player, when moving at a given stage, cannot exclude he is moving from, constitutes the information set of that player at that stage of the game.

• When information sets are all singleton (made of a single node) the game is one of perfect information.

• In games of imperfect information some information sets includes more than one node.
Prisoner dilemma in extensive form

- The graphical convention allows us express simultaneous (static) games in extensive form
- This is the correspondence for the PD game

<table>
<thead>
<tr>
<th></th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>cooperate</td>
</tr>
<tr>
<td>A</td>
<td>cooperate</td>
</tr>
<tr>
<td></td>
<td>defect</td>
</tr>
</tbody>
</table>

- The game is symmetric: rows and columns can be exchanged in normal form
- First and second move can be exchanged in extensive form
Imperfect information and backward induction

In games of imperfect information **backward induction** procedure has to be considered carefully

⇒ When a player does not know which node he is moving from he **cannot single out** the best action

⇒ The preceding player into the game tree **cannot anticipate** what he will do...unless one special case applies...
A special case: prisoner dilemma

Even if B does not know if he is in $B_S$ or in $B_I$ he can observe that, whatever his starting node, “defect” gives him more utility than “cooperate”:

• 4 instead of 3 if A has done “cooperate”
• 2 instead of 1 if A has chosen “defect”

This is because “cooperate” is a dominated strategy
The general case

• Let us go back to the “Challenge into the party” game

<table>
<thead>
<tr>
<th>Y Choice (ch)</th>
<th>O Choice (at)</th>
<th>Outcome</th>
</tr>
</thead>
<tbody>
<tr>
<td>Co-opt (co)</td>
<td>(0,1) crisis</td>
<td></td>
</tr>
<tr>
<td>Attack (at)</td>
<td>(2,3) diarchy</td>
<td></td>
</tr>
<tr>
<td>Rest. (co)</td>
<td>(1,5) restoration</td>
<td></td>
</tr>
<tr>
<td>Appeas. (co)</td>
<td>(3,4) appeasement</td>
<td></td>
</tr>
</tbody>
</table>

• If Y chooses “challenge”, O will choose “co-opt” as he prefers sharing the power (diarchy) to a party fission (crisis)
• If Y chooses “no challenge”, O will choose “attack” as he prefers regain his absolute power (restoration) to an agreement with his younger competitor (appeasement)
• Y cannot anticipate O choice and backward induction reasoning cannot start
Subgames with imperfect information

A subgame is a subset of the extensive form that satisfies the following criteria:

1. It begins at a node (singleton)
2. It includes all nodes following this initial node and no others
3. It does not cut any information sets (if two nodes are part of the same information set they belong to the same subgame)
Subgames with imperfect information

• Take any node $x$ in an extensive form. Examine the collection of nodes given by $x$ and its successors. If there is a node $y$ that is not a successor of $x$ but it is connected to $x$ or one of its successors by a dashed line, then $x$ does not initiate a subgame.

• In other words, once players are inside a subgame, it is common knowledge between them that they are inside it.
Subgames with imperfect information

• Subgames are self-contained extensive forms, meaningful trees on their own
• Subgames that start from nodes other than the initial node are called proper subgames
• Of course in a game of perfect information, every node initiates a subgame!
Information sets and subgames (1)

- Player A has one information set that is a singleton
- Player B has one information set with two nodes (none of which is a singleton)
- No proper subgames exist for PD but the whole game
Information sets and subgames (2)

Each player A, B, C has three moves:
\{U, M, D\} ; \{u, m, d\} ; \{u, \mu, \delta\}

- At the first stage **A has one information set** (singleton)
- At the second stage B knows that A either chooses \{D\} or \{U,M\} that are the **two information sets of B**
- At the third stage C knows that B has chosen among \{Dd,Dm\}, \{Du\}, \{Md,Mm,Ud,Um\}, \{Mu,Uu\} that are **four C’s information sets**

The game has **three subgames (two proper subgames)**: the original game, a subgame following D, a subgame following the path D-u
Role of information sets

• Many real political or social events may present phases of both **simultaneity** and **time dependent interaction**

• Through the concept of information set game theory can face strategic interactions that are partly sequential and partly simultaneous (which is the same as interactions made partly by visible and partly by hidden moves)
A meaningless information set
A conventionally excluded information set

This game is admissible only if player A in $A_2$ has forgotten what she has done in $A_1$
We assume that all players remember their past actions
Games are of **perfect recall**
Strategies with imperfect information

• To determine NE it is necessary to know the strategies of each player \( i \) (\( i = 1, 2, \ldots, n \))

• A definition of strategy (a complete plan of action) is needed for games in extensive form

• For any given player \( i \) a strategy specifies what that player should do at any information set

• How to solve such game?
Example: strategies in the tree

By definition
- A has four strategies:
  - \{UV, UE, DV, DE\}
- B has two strategies:
  - \{u, d\}
- Although DV and DE are sequences of moves never played by A, they cannot be discarded: credibility of threat!
From the tree to the matrix ...

The matrix shows three NE: $\{UV,u\}, \{DV,d\}, \{DE,d\}$
Normal form and extensive form

• At first sight controlling the strategies seems the same as putting the game in normal form
• However not all equilibria in normal form satisfy backward induction
• Taking into account the sequence of moves, game trees (extensive form) give information on games that matrices (normal form) do not give
• Trees and information sets give a more precise account than matrices of social interactions
... and back to the tree (1)

- NE: \{UV,u\}, \{DV,d\}, \{DE,d\}
- Are all those equilibria coherent with backward induction?
  - By backward induction if A has the opportunity to make her second move, she will choose E (2>1)
  - \{UV,d\} and \{DV, d\} do not therefore satisfy subgame perfection!!!
  - The only strategy profile coherent with backward induction is \{DE,d\} \(\Rightarrow (3,4)\)
... and back to the tree (2)

- NE: \{UV,u\}, \{DV,d\}, \{DE,d\}
- The same result is more quickly achievable through the concept of subgames

There are two subgames
1. The first starts at the node where A moves for the second time
2. The second is the whole game

The only SPNE is \{DE,d\} \rightarrow (3,4)
A further example: three players

How can we represent a three players game in normal form?
## Two matrices

<table>
<thead>
<tr>
<th>A plays U</th>
<th>A plays D</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>C</strong></td>
<td><strong>C</strong></td>
</tr>
<tr>
<td></td>
<td>left</td>
</tr>
<tr>
<td><strong>B</strong></td>
<td>u</td>
</tr>
<tr>
<td></td>
<td>d</td>
</tr>
</tbody>
</table>

A: matrix player  
B: row player  
C: column player

The game has various NE, which ones?  
\{U,u,left\},  
\{U,u,right\},  
\{U,d,left\},  
\{D,d,right\}  
For instance: \{U,u,left\} leading to (3,1,1) is one

Is it a SPNE?
Consider the subgame where only players B and C play

\[
\begin{array}{c|cc}
& \text{left} & \text{right} \\
\hline
\text{u} & 2,2 & 0,3 \\
\text{d} & 0,0 & 2,1 \\
\end{array}
\]

“right” is a dominant strategy for player C and the matrix of the subgame has the only NE \{d, right\}.

That excludes \{U, u, left\} from being a SPNE.

The only SPNE of the original game is \{D, d, right\} leading to the outcome (4,2,1).

Procedure to find out SPNE: first solve subgames and then go back to the whole game or vice versa. It should always produce the same result!
Let’s discuss some more examples